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Math. 243

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Test 1 — Spring Semester 2014/2015

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Q1 (20 points) Answer the following statements by true or false:

T (1) If A is a singular matrix then A^t is singular is a proposition.

F (2) I am happy is a proposition.

T (3) If the square system $AX = b$ has more than one solution then A is nonsingular is a proposition.

T (4) The truth set of: $|x| = 2$ is -2, 2.

F (5) The truth set of: $\frac{1}{x} < 1$ is all real numbers greater than 1. $x>0$ $x>1$ $x<0$ $x<0$

T (6) $p\vee \sim p$ is a tautology.

T (7) $p\wedge \sim p$ is a contradiction.

T (8) $p \to q \leftrightarrow \sim q \to \sim p$ is a tautology.

T (9) $x^7 + 3x^4 + x + 2$ has no positive real solution $=0$

F (10) For any set A $A \subseteq P(A)$. $\in$

F (11) If A, B are disjoint then A^c, B^c are disjoint.

F (12) If $A \subseteq B$ then $A^c \subseteq B^c$.

T (13) $A - B = A \cap B^c$.

F (14) $(A \cap B)^c = A^c \cap B^c$.

T (15) $\phi \in \{\phi\}$.

T (16) ϕ is an inductive set.

T (17) The only an inductive set that contains 1 is N.

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$k \in R$, $k \geq 4$

F (18) The set of all real numbers that are greater than or equal to 4 is an inductive set. $\subseteq N$

F (19) Any set that contains an inductive set is an inductive set. A $P(A)$

F (20) Any subset of an inductive set is an inductive set.

Q2 (10 points) Negate each of the following statements:

(1) (x is an integer which is a perfect square)

x is an Integer which is not a perfect square $\equiv$ [x is an Integer But not a perfect square]

(2) $\sim [z \wedge (x \vee y)] \equiv \sim z \vee ((\sim x) \wedge (\sim y))$

(3) $\sim (x \wedge y) \equiv (\sim x) \vee (\sim y)$

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(4) 3 is a prime number $\rightarrow$ 3 is not a prime number.

(5) If $A \subseteq B$ then B is nonempty $\rightarrow$ $A \subseteq B$ and B is empty.

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Q3 (35 points) Prove or disprove each of the following:

(1) $\sim (x \vee y)$ is $\sim (x) \wedge (\sim y)$

$\Rightarrow$ Suppose $\sim (x \vee y)$ is True. So $(x \vee y)$ is False. So x And y are both False

So $(\sim x)$ And $(\sim y)$ both are True. So $\overset{T}{(\sim x)} \overset{\wedge}{\wedge} \overset{T}{(\sim y)}$ is True.

$\Leftarrow$ Suppose $(\sim x) \wedge (\sim y)$ is True. So $(\sim x)$ And $(\sim y)$ both are True.

So (x) And (y) are False So $(x \vee y)$ is False. So $\sim (x \vee y)$ is True.

T (2) $\sim [(\forall x)(p(x))]$ is $(\exists x)(\sim p(x))$

$\Rightarrow$ Suppose $\sim [(\forall x)(p(x))]$ is True. So $[(\forall x)(p(x))]$ is False.

So the Truth set is not the universal set. So There exist x such that not $p(x)$ $\longrightarrow$ $(\exists x)(\sim p(x))$ is True.

$\Leftarrow$ Suppose $[(\exists x)(\sim p(x))]$ is True. So [for all x such that $p(x)$ is True] is ~~False~~ $(\forall x)(p(x))$ is False

So $\sim [(\forall x)\ p(x)]$ is True.

So LHS = RHS.

(3) If a, b, c positive integers such that a divides bc, then a divides b or a divides c False.

Counter Example:- (If $bc = ak$, $k \in \mathbb{Z} \rightarrow b = ak_2$ or $c = ak_3$) $k_1, k_2, k_3 \in \mathbb{Z}$

$b = 3$
$c = 6$
$a = 9$

let $bc = ak$

$3 \times 6 = 9k$

$18 = 9(k)$

$18 = 9(2)$ ✓

but $b \neq ak_2$

$3 \neq 9k_2$ since $k_2 \in \mathbb{Z}$

$\nexists k_2 \in \mathbb{Z}$ such that $3 = 9k_2$.

$c = ak_3$

$6 \neq 9k_3$ since $k_3 \in \mathbb{Z}$ $\nexists k_3 \in \mathbb{Z}$ such that $6 = 9k_3$

(4) If an integer a^2 is even then a is even.

Contrapositive: If a is odd Then a^2 is odd.

Suppose $a \in \mathbb{Z}$ ~~And~~ a is odd. [need to show a^2 is odd.] So there exist $k \in \mathbb{Z}$ such that $a = 2k+1$ (a is odd)

$a^2 = (2k+1)^2 = 4k^2 + 4k + 1 = 2(2k^2 + 2k) + 1 = 2c + 1$ which is odd.

$c = (2k^2 + 2k)$; $c \in \mathbb{Z}$.

(5) If $\frac{1}{x} < \frac{1}{2}$ then $x > 2$. False..

~~Suppose $\frac{1}{x} < \frac{1}{2}$, need to show $x > 2$.~~

Suppose $\frac{1}{x} < \frac{1}{2}$, need to show $x > 2$

~~Take $x = \frac{3}{2}$~~

Case 1: If $x > 0$

$\frac{1}{x} < \frac{1}{2}$

$2 < x$

Then $\boxed{x > 2}$

Case 2: If $x < 0$

$\frac{1}{x} < \frac{1}{2}$

$2 > x$

Then $\boxed{x < 0}$

So The Statement is False.

Take $x = -1$

$\frac{1}{x} < \frac{1}{2}$

$\frac{-1}{1} < \frac{1}{2}$ ✓

(6) $\sqrt{2}$ is irrational

Assume not. $\rightarrow$ [$\sqrt{2}$ is rational]

So $\exists\, a, b \in \mathbb{Z}$, $b \neq 0$ such that $\sqrt{2} = \frac{a}{b}$ And $\frac{a}{b}$ is in the simple form.

$(\sqrt{2})^2 = \left(\frac{a}{b}\right)^2$

$2 = \frac{a^2}{b^2}$

$2b^2 = a^2$ So a^2 is even

$\boxed{\text{So } a \text{ is even.}}$

So $a = 2k$; $k \in \mathbb{Z}$

$a^2 = 4k^2$

$2 = \frac{4k^2}{b^2}$

$2b^2 = 4k^2$

$b^2 = 2k^2$

So b^2 is even

$\boxed{\text{So } b \text{ is even}}$

If a is even & b is even So a, b have common divisor $\{2\}$

So $\frac{a}{b}$ is not in the simple form ✗.

$\Rightarrow \sqrt{2}$ is irrational (True)

(7) Let $p_1, p_2, ..., p_n$ be distinct prime numbers. Show that $p_1p_2...p_n + 1$ is not divisible by any p_i, $i = 1, ..., n$

By Contradiction

Suppose $P_1, P_2, ..., P_n$ is distinct prime But $(P_1P_2 \cdots P_n)+1$ is divisible by P_i

$(P_1P_2 \cdots P_n)$ is divisible by P_i, Since P_i is one of them.
So P_i divides the difference.

$(P_1 P_2 \cdots P_n)+1 - (P_1P_2 \cdots P_n) = 1$ ~~Since 1 is not~~

(8) Let n be a positive integer. Show that n is either a prime number, or a perfect square or $(n-1)!$ is divisible by n

Suppose n a positive Integer And not a prime number And not perfect square And $(n-1)!$ is divisible by n.

Let $n = st$ such that $s \neq t$ [Since n is not perfect square] and $1 < s, t < n$ (Since n not prime)

$(n-1)! = (n-1)\, n \cdots (3)(2)(1)$.

$= (n-1) \cdots (3)(2)(1)$ So s, t divides $(n-1)!$

s, t

So n divides $(n-1)!$

$\Rightarrow (n-1)!$ is divisible by n.

Q4 (40 points) $\boxed{\text{Prove}}$ each of the following:

(1) For any sets A, B, $A \subseteq B$ iff $P(A) \subseteq P(B)$

$(\Rightarrow)$ If $A \subseteq B$, then $P(A) \subseteq P(B)$. Suppose $A \subseteq B$, need to show $P(A) \subseteq P(B)$.
Let $X \in P(A)$ So $X \subseteq A$ Since $A \subseteq B$ So $X \subseteq B$ So $X \in P(B)$.

$(\Leftarrow)$ If $P(A) \subseteq P(B)$, then $A \subseteq B$. Suppose $P(A) \subseteq P(B)$, need to show $A \subseteq B$.
Let $x \in A$ So $\{x\} \in P(A)$ Since $P(A) \subseteq P(B)$ So $\{x\} \in P(B)$ So $x \in B$

(2) If A, B two sets such that $A \cap B = \phi$. Then $A \subseteq B^c$.

Suppose $A \cap B = \phi$.

Let $x \in A$ Since $A \cap B = \phi$ So $x \notin B$. So $x \in B^c$ So $A \subseteq B^c$.

(3) $A - B = A \cap B^c$.

Let $x \in (A-B) \to x \in (A \cap B^c) \to x \in A$ And $x \in B^c$

$\to x \in (A \cap B^c)$. LHS $\subseteq$ RHS.

Let $x \in A \cap B^c \to x \in A$ And $x \in B^c \to x \in A$ And $x \notin B$

RHS $\subseteq$ LHS $\to x \in (A-B)$.

(4) $(\bigcap_{n \in N} A_n)^c = \bigcup_{n \in N} A_n^c$

So LHS = RHS.

$\overline{\bigcap_{n \in N} A_n} = \bigcup_{n \in N} A^c$

* Suppose $x \in \overline{\bigcap_{n \in N} A_n} \to x \notin (\bigcap_{n \in N} A_n) \to x \notin (A_1 \cap A_2 \cap \ldots \cap A_n \ldots)$

$x \in (A_1 \cap A_2 \ldots \cap A_n)^c \to x \in (\overline{A_1} \cup \overline{A_2} \ldots \cup \overline{A_n} \ldots)$

So $x \in \bigcup_{n \in N} A^c$ So LHS $\subseteq$ RHS.

* Suppose $x \in \bigcup_{n \in N} A^c \to x \notin (\bigcup_{n \in N} A^c)^c \to \cancel{x \notin (\overline{A_1} \cup \overline{A_2} \cup \ldots \overline{A_n} \cup \ldots)^c}$

1 $x \notin (A_1 \cap A_2 \ldots \cap A_n) \to x \in (A_1 \cap A_2 \ldots \cap A_n)^c \to x \in (\bigcap_{n \in N} A_n)^c$

So RHS $\subseteq$ LHS.

So LHS = RHS.

(5) If $A \subseteq B$, then $B^c \subseteq A^c$. (Assume $A \subseteq B$, need to show $B^c \subseteq A^c$)

Assume $x \in B^c$ So $x \notin B$ Since $A \subseteq B$ So $x \notin A$

So $x \in A^c$.

$\Rightarrow B^c \subseteq A^c$.

$n^3 - n = 6k \quad (k \in \mathbb{Z})$

(6) Show that for any positive integer n, $n^3 - n$ is divisible by 6

Prove it By First Principle of Mathematical Induction

1) Show it is True at $n=1$

$LHS = n^3 - n = (1)^3 - (1) = 0 \qquad 0 = 6k$ ✓ $(k=0 \in \mathbb{Z})$

2) Suppos it is True at $n=k$. So k^3-k is divisible by 6.

So $k^3 - k = 6c \quad (c \in \mathbb{Z})$.

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3) Prove it is True at $n=k+1$.

Show $(k+1)^3 - (k+1)$ is disible by 6.

$(k+1)^3 - (k+1) = (k+1)\left[(k+1)^2 - 1\right] = (k+1)\left[k^2+2k\right] = k^3 + 2k^2 + k^2 + 2k$

$k^3 + 3k^2 + 2k = \underbrace{k^3 - k}_{\text{from step 2}} + 3k^2 + 3k = 6c + 3(k^2+k) = 6c + 3k(k^2+1)$

(7) Use the division algorithm and the proof by cases to show that $n^3 - n$ is divisible by 3

$n^3 - n = 3q + r \qquad 0 \le r < 3$

$3 \overline{) n^3 - n}$ q, r

Case 1: $r=0$

$n^3 - n = 3q \quad (q \in \mathbb{Z})$ So $n^3 - n$ is divisibl by 3. ✓

Case 2: $r=1$

$n^3 - n = 3q + 1 \quad (q \in \mathbb{Z}) \rightarrow$ is divisible by 3.

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Case 3: $r=2$

$n^3 - n = 3q + 2 \quad (q \in \mathbb{Z}) \rightarrow$ is divisible by 3.

(8) Let $A_n = (\frac{-1}{n}, \frac{1}{n})$, $n \in N$. Show that $\bigcap_{n \in N} A_n = \{0\}$

$x \notin A_n \implies x \notin \bigcap_{n=1}^{\infty} A_n$

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So $\forall x \in (-1,1), x \neq 0 \quad x \notin \bigcap_{n=1}^{\infty} A_n$

So $\bigcap_{n=1}^{\infty} A_n = \{0\}$.

If $-1 < y$, then $x = -y > 0$

So $\exists N \in \mathbb{Z}^+$ such that $\frac{1}{N} < x$

$y \notin A_n \implies y \notin \bigcap_{n=1}^{\infty} A_n$

$A_1 \supseteq A_2 \supseteq A_3 \cdots \supseteq A_n$

(9) Show that Archemedian property implies that for any positive real numbers a, b there exists a natural number N such that $a < bN$.

Archamedian · For any $x \in R$, $x > 0$ $\exists N$ such that $\frac{1}{N} < x$.

· $\forall (a,b) \in R$, $(\exists N \in N) ; (a < bN)$

$bN - a = \mathcal{E}$

Suppose $b = \frac{1}{a} > 0$, $N = [a] + 3$

$\frac{1}{N} < \frac{1}{a}$

$\frac{1}{N} < b$

(10) Show that if for any positive real numbers a, b there exists a natural number N such that $a < bN$ then Archemedian property holds (for any $\epsilon > 0$ there exists a natural number N such that $\frac{1}{N} < \epsilon$).

Assume $a < bN$ need to show $(\forall \mathcal{E} > 0)(\exists N \in N) ; (\frac{1}{N} < \epsilon)$

$bN - a = \mathcal{E}$

$\mathcal{E} > 0$ Suppose $y = \frac{1}{\mathcal{E}} > 0$

$N = [y] + 3 > x$

$\frac{1}{N} < \frac{1}{y}$

So $\frac{1}{N} < \mathcal{E}$